First results of hybrid HKM for RHIC and LHC energies

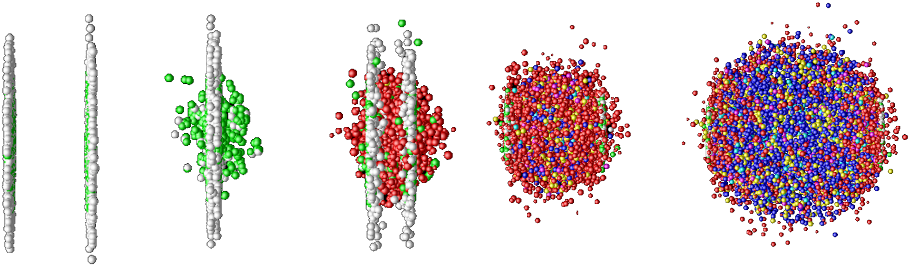
Yuri KARPENKO

In collaboration with Yuri SINYUKOV and Klaus WERNER

Bogolyubov Institute for Theoretical Physics, NAS of Ukraine

XI GDRE, Nantes

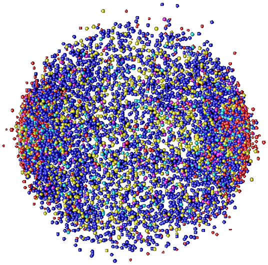
Introduction: heavy ion collision in pictures[1]



- Initial state, hard scatterings
- Thermalization
- Hydrodynamic expansion
  - Quark-gluon plasma
  - Phase transition
  - Hadron Gas
  - Chemical freeze-out
- Kinetic stage
- (kinetic) freeze-out

Typical size
10 fm $\propto 10^{-14}$m

Typical lifetime
10 fm/c $\propto 10^{-23}$s



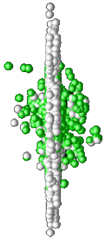
[1] taken from event generator

Hybrid Hydro-Kinetic Model

Ingredients:

- Initial conditions (Glauber model)
- Hydrodynamic solution
  - Equation of state for hydrodynamics
- Hydro-kinetic approach: deviations from local equlibrium
- Boltzmann cascade (UrQMD)

Thermally equilibrated evolution



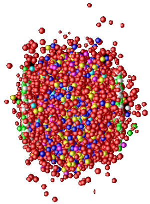
Initial conditions at $\tau_0 = 1$fm/c. "Effective" initial distribution, bringing average hydrodynamic results for EbE case.

- Glauber model

$$\varepsilon(\mathbf{b},\mathbf{r_T}) = \varepsilon_0 \frac{\rho(\mathbf{b},\mathbf{r_T})}{\rho_0}$$

$$\rho(\mathbf{b},\mathbf{r_T}) = T(\mathbf{r_T}-\mathbf{b}/2)S(\mathbf{r_T}+\mathbf{b}/2) + T(\mathbf{r_T}+\mathbf{b}/2)S(\mathbf{r_T}-\mathbf{b}/2)$$

Rapidity profiles: $y_T = \alpha\frac{r_T}{R_T}$ (nonzero initial flow), $y_L = \eta$ (boost-inv.)
ε_0 and α are the only fitting parameters in the model.



Hydrodynamic approach

$$\partial_\nu T^{\mu\nu} = 0$$

$$\partial_\mu(n_i \cdot u^\mu) = 0$$

ideal fluid:

$$T^{\mu\nu} = (\varepsilon+p)u^\mu u^\nu - p\cdot g^{\mu\nu}$$

+equation of state $p = p(\varepsilon,\{n_i\})$

$i = B,E,S$ in QGP phase
$i = 1\ldots N$, $N = 329$ in hadron phase (see EoS, below)

Hydrodynamics

- Bjorken(light-cone in z-direction) coordinates :

$$\tau = (t^2 - z^2)^{1/2}, \quad \eta = \frac{1}{2} \ln \frac{t+z}{t-z}$$

- Conservative variables:

$$\vec{Q} = \begin{pmatrix} Q_\tau \\ Q_x \\ Q_y \\ Q_\eta \\ \{Q_{n_i}\} \end{pmatrix} = \begin{pmatrix} \gamma^2(\varepsilon + p) - p \\ \gamma^2(\varepsilon + p) v_x \\ \gamma^2(\varepsilon + p) v_y \\ \gamma^2(\varepsilon + p) v_\eta \\ \{\gamma n_i\} \end{pmatrix}$$

- Velocity transformation:

$$v_x = v_x^{lab} \cdot \frac{\cosh y_f}{\cosh(y_f - \eta)}$$
$$v_y = v_y^{lab} \cdot \frac{\cosh y_f}{\cosh(y_f - \eta)}$$
$$v_\eta = \tanh(y_f - \eta) \qquad (1)$$

- Hydrodynamic equations:

$$\partial_\tau \underbrace{\begin{pmatrix} Q_\tau \\ Q_x \\ Q_y \\ Q_\eta \\ \{Q_{n_i}\} \end{pmatrix}}_{\text{quantities}} + \underbrace{\vec{\nabla} \cdot \begin{pmatrix} Q_\tau \\ Q_x \\ Q_y \\ Q_\eta \\ \{Q_{n_i}\} \end{pmatrix} \vec{v} + \begin{pmatrix} \vec{\nabla}(p \cdot \vec{v}) \\ \partial_x p \\ \partial_y p \\ \frac{1}{\tau}\partial_\eta p \\ 0 \end{pmatrix}}_{\text{fluxes}} + \underbrace{\begin{pmatrix} (Q_\tau + p)(1 + v_\eta^2)/\tau \\ Q_x/\tau \\ Q_y/\tau \\ 2Q_\eta/\tau \\ \{Q_{n_i}/\tau\} \end{pmatrix}}_{\text{sources}} = 0$$

where $\vec{\nabla} = \left(\partial_x,\ \partial_y,\ \frac{1}{\tau}\partial_\eta\right)$

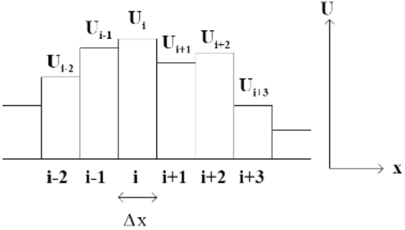
Hydrodynamics: basic method

For central A+A collisions and midrapidity: suppose longitudinal boost-ivariance and axial symmetry in transverse plane. Thus, $Q_\phi = Q_\eta = 0$, and flows $F_\phi = F_\eta = 0$.

$$\partial_\tau \underbrace{\begin{pmatrix} Q_\tau \\ Q_r \\ \{Q_{n_i}\} \end{pmatrix}}_{\text{quantities}} + \partial_r \cdot \underbrace{\begin{pmatrix} (Q_\tau + p)v_r \\ Q_r v_r + p \\ \{Q_{n_i} v_r\} \end{pmatrix}}_{\text{fluxes}} + \underbrace{\begin{pmatrix} (Q_\tau + p)(1+v_\eta^2)/\tau - (Q_\tau + p)v_r/r \\ Q_r/\tau - Q_r v_r/r \\ \{Q_{n_i}/\tau - Q_{n_i} v_r/r\} \end{pmatrix}}_{\text{sources}} = 0 \qquad (2)$$

Finite volume method:

- Divide the space into cells, Q_i in the average value of quantity inside cell



- The numerical equations:

$$Q^{n+1}_{ijk} = Q^n_{ijk} - \frac{\Delta t}{\Delta x_1}(F_{i+1/2,jk} + F_{i-1/2,jk}) - \frac{\Delta t}{\Delta x_2}(F_{i,j+1/2,k} + F_{i,j-1/2,k}) - \frac{\Delta t}{\Delta x_3}(F_{ij,k+1/2} + F_{ij,k-1/2})$$

F - time-averaged flow through the cell interface.

Hydrodynamics: numerical algorithm

- Flow through the cell interface depends only on the Riemann problem solution for this interface (+CFL condition)
- We use rHLLE solver for Riemann problem
- *predictor-corrector* scheme is used for the second order of accuracy in time, i.e. the numerical error is $O(dt^3)$, instead of $O(dt^2)$
- in space : in the same way, to achieve the second order scheme the *linear distributions* of quantities (conservative variables) inside cells are used.
- *Multi-dimension problem*: we use the metod, similar to operator(dimensional) splitting, but symmetric in all dimensions.
- *Grid boudaries*: we use the method of *ghost cells*, outflow boundary.
- *Vacuum treatment*: since initial grid covers both system and surrunding vacuum, we account for finite velocity of expansion into vacuum.

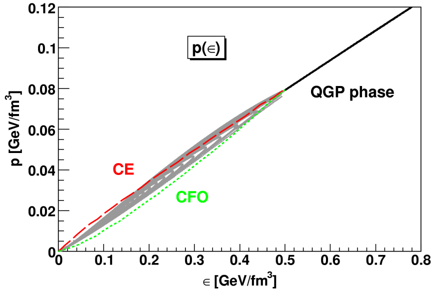
Equation of state

Equation of state, QGP, $T > T_c$ Realistic equation of state[2], consistent with lattice QCD results with crossover-type phase transition at $T_c = 175$ MeV, transforming into multicomponent hadron gas at $T = T_c$ ($\mu_B = 0$).
To account for chagre conservation in QGP phase $\rightarrow$ corrections for nonzero μ_B, μ_S [3]:

$$\frac{p(T,\mu_B,\mu_S)}{T^4} = \frac{p(T,0,0)}{T^4} + \frac{1}{2}\frac{\chi_B}{T^2}\left(\frac{\mu_B}{T}\right)^2 + \frac{1}{2}\frac{\chi_S}{T^2}\left(\frac{\mu_S}{T}\right)^2 + \frac{\chi_{BS}}{T^2}\frac{\mu_B}{T}\frac{\mu_S}{T} \qquad (3)$$

Expansion coefficients χ_B, χ_S are baryon and strangs susceptibilies.

$$\frac{\mu_\alpha}{T} = const_\alpha, \quad \alpha = B, Q, S$$



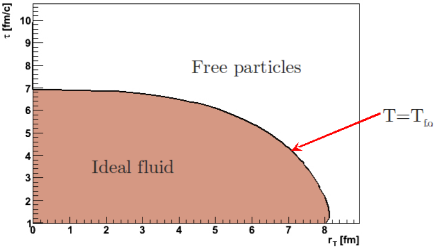
Chemical freeze-out at $T_{ch} = 165 MeV$, corresponding $\mu_B = 29 MeV$, $\mu_S = 7 MeV$, $\mu_Q = -1 MeV$ and $\gamma_S = 0.935$ suppression factor, dictated by particle number ratios analysis at 200A GeV RHIC.
Hadron gas at $T < T_{ch}$. N=359 particle number densities are introduced, corresponding to each sort of hadrons. Yields from resonance decays are effectively included (massive resonance approximation):

$$\partial_\mu (n_i u^\mu) = -\Gamma_i n_i + \sum_j b_{ij} \Gamma_j n_j$$

Grey points: differen chemical compositions

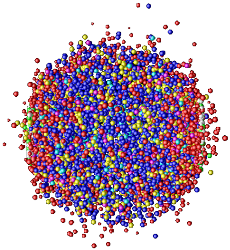
[2] M. Laine, Y. Schroder Phys. Rev. D73 (2006) 085009.
[3] F. Karsch, PoS CPOD07:026, 2007.

Final stage of evolution



Connecting hydrodynamic and kinetic(final) stage: Cooper-Frye prescription

Problems on non-space-like sectors of switching hypersurface...



Final stage (weakly interacting system)

- UrQMD (afterburner)
  C. Nonaka, S.A. Bass
- JAM (afterburner)
  T. Hirano, M. Gyulassy
- THERMINATOR
  W. Florkowski, W. Broniowski, M. Chojnacki, A. Kisiel
- FASTMC
  N.S. Amelin, R. Lednicky, T.A. Pocheptsov, I.P. Lokhtin, L.V. Malinina, A.M. Snigirev, Iu.A. Karpenko, Yu.M. Sinyukov

Hydro-kinetic approach

Goal: to connect hydrodynamic and final stages in a more natural way (not Cooper-Frye prescription), and account for the deviations from local equilibrium at hydrodynamic stage (remember viscosity?).

- is based on relaxation time approximation for emission function of relativistic finite expanding system

- provides evaluation of escape probabilities and deviations of distribution functions from local equilibrium
  Zero approximation is ideal hydro.

Complete algorithm:

- solution of equations of ideal hydro
- calculation of non-equilibrium DF and emission function in first approximation
- solution of equations for ideal hydro with non-zero left-hand-side that accounts for conservation laws for non-equilibrium process of the system which radiated free particles during expansion
- Calculation of “exact” DF and emission function
- Evaluation of spectra and correlations / **input to UrQMD**

Yu.M. Sinyukov, S.V. Akkelin, Y. Hama, Phys.Rev.Lett.89:052301,2002
S.V. Akkelin, Y. Hama, Iu.A. Karpenko, Yu.M. Sinyukov, Phys.Rev.C78:034906, 2008

Hydrokinetic approach to particle emission

Particle liberation from hydrodynamically expanding system is described by the approximate method inspired by the integral form of Boltzmann equation [4]:

$$f_i(t,\vec{x},p) = f_i(\bar{x}_{t\to t_0},p)\mathscr{P}_{t_0\to t}(\bar{x}_{t\to t_0},p) + \int_{t_0}^{t} \underbrace{G_i((\bar{x}_{t\to s},p))\mathscr{P}_{s\to t}(\bar{x}_{t\to s},p)}_{S(\bar{x}_{t\to s},p)} ds, \qquad \bar{x}_{t\to s} = (s,\vec{x}+\frac{\vec{p}}{p^0}(s-t))$$

where $\frac{p^\mu}{p^0}\frac{\partial f_i(x,p)}{\partial x^\mu} = G_i(x,p) - L_i(x,p)$, $L_i(x,p) = R_i(x,p)f_i(x,p)$ and $\mathscr{P}_{t\to t'}(x,p) = \exp\left(-\int_t^{t'} d\bar{t} R_i(\bar{x}_t,p)\right)$

Relaxation time approximation for collision terms (if i=stable particle):

$$R_i(x,p) \approx R_i^{l.eq.}(x,p) = \text{collision rate}, \quad \text{and} \quad G_i \approx R_i^{l.eq.}(x,p)f_i^{l.eq.}(x,p) + G_i^{\text{decay}}(x,p)$$

$$\Downarrow$$

$$\frac{p^\mu}{p_0}\frac{\partial f_i(x,p)}{\partial x^\mu} = -\frac{f_i(x,p) - f_i^{\text{l.eq.}}(x,p)}{\tau_{\text{rel}}(x,p)} + G_i^{\text{decay}}(x,p).$$

First approximation (ideal hydro!):

$$\partial_\nu T_i^{\nu\mu}[f_i^{\text{l eq}}] = 0, \quad \partial_\nu n^\nu[f_i^{\text{l eq}}] = 0$$

The "relaxation time" $\tau_{rel} = 1/R_i^{l.eq.}$ grows with time!

For i-th coomponent of hadron gas, in Bjorken coordinates:

Emission function

$$S_i(\lambda,\theta,r_T,p) = \left[f_i^{l.eq.}(\lambda,\theta,r_T,p)\tilde{R}_i(\lambda,\theta,r_T,p) + \tilde{G}_i^{decay}(\lambda,\theta,r_T,p)\right] \exp\left(-\int_\lambda^\infty \tilde{R}_i(s,\theta^{(s)}(\lambda),r_T^{(s)}(\lambda),p)ds\right)$$

[4] for the details, see Yu.M. Sinyukov, S.V. Akkelin, Y. Hama, Phys.Rev.Lett.89:052301,2002

$$p_i^0 G_i^{decay}(x,p_i) = \sum_j \sum_k \int \frac{d^3 p_j}{p_j^0} \int \frac{d^3 p_k}{p_k^0} \Gamma_{j\to ik} f_j(x,p_j) \frac{m_j}{F_{j\to ik}} \delta^{(4)}(p_j - p_k - p_i)$$

Collision rate (inverse relaxation time) for *i*-th sort of hadrons:

$$\frac{1}{\tau^{\mathrm{id}*}_{i,\mathrm{rel}}(x,p)} = R_i^{\mathrm{id}}(x,p) = \int \frac{d^3 k}{(2\pi)^3} \exp\left(-\frac{E_k - \mu_{id}(x)}{T_{id}(x)}\right) \sigma_i(s) \frac{\sqrt{s(s-4m^2)}}{2E_p E_k}.$$

where $E_p = \sqrt{\mathbf{p}^2 + m^2}$, $E_k = \sqrt{\mathbf{k}^2 + m^2}$, $s = (p+k)^2$ is squared pair energy in CMS, $\sigma(s)$ - cross-section, calculated in a way similar to URQMD. Observable quantity: particle spectrum,

$$\frac{d^3 N_i}{d^3 p} = n_i(p) = \int_{t\to\infty} d^3 x \; f_i(t,x,p)$$

Kinetics: inverse relaxation time (collision rate)

collision rate (inverse relaxation time):

$$\frac{1}{\tau_{\rm rel}^{\rm id*}(x,p)} = R^{\rm id}(x,p) = \int \frac{d^3k}{(2\pi)^3} \exp\left(-\frac{E_k - \mu_{id}(x)}{T_{id}(x)}\right) \sigma(s) \frac{\sqrt{s(s-4m^2)}}{2E_p E_k}.$$

where $E_p = \sqrt{\mathbf{p}^2 + m^2}$, $E_k = \sqrt{\mathbf{k}^2 + m^2}$, $s = (p+k)^2$ is squared pair energy in CMS, $\sigma(s)$ - cross-section, calculated in a way similar to URQMD.:

- meson-meson, meson-baryon:

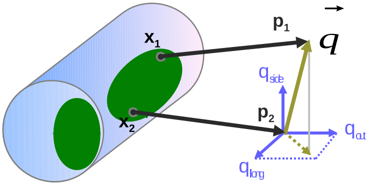
$$\sigma_{tot}^{MB}(\sqrt{s}) = \sum_{R=\Delta,N^*} \langle j_B, m_B, j_M, m_M \| J_R, M_R \rangle \frac{2S_R+1}{(2S_B+1)(2S_M+1)} \times \frac{\pi}{p_{cm}^2} \frac{\Gamma_{R\to MB}\Gamma_{tot}}{(M_R - \sqrt{s})^2 + \Gamma_{tot}^2/4} \quad ,$$

  +5 mbarn for elastic meson-meson scattering
- $p-p$, $p-n$, $p-\bar{p}$, etc. $\to\to$ tables
- other: additive quark model:

$$\sigma_{\rm total} = 40\left(\frac{2}{3}\right)^{m_1+m_2}\left(1-0.4\frac{s_1}{3-m_1}\right)\left(1-0.4\frac{s_2}{3-m_2}\right)[\rm mb] \quad ,$$

  $m_i = 1(0)$ for meson (baryon), s_i - number of strange quarks in particle i.

HBT(interferometry) measurements



$$\vec{q} = \vec{p}_2 - \vec{p}_1$$

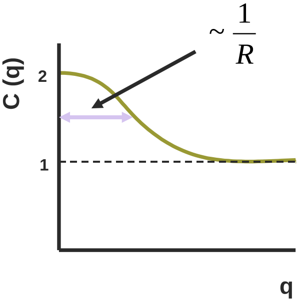
$$\vec{k} = \frac{1}{2}(\vec{p}_1 + \vec{p}_2)$$

$$C(p_1, p_2) = \frac{P(p_1, p_2)}{P(p_1)P(p_2)} = \frac{\text{real event pairs}}{\text{mixed event pairs}}$$

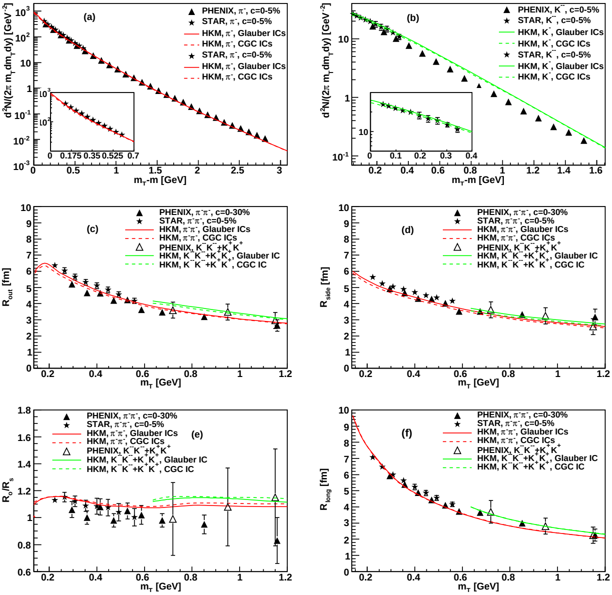
Gaussian approximation of CFs
($q \to 0$):

$$C(\vec{k}, \vec{q}) = 1 + \lambda(k) e^{-q_{out}^2 R_{out}^2 - q_{side}^2 R_{side}^2 - q_{long}^2 R_{long}^2}$$

$R_{out}, R_{side}, R_{long}$ (HBT radii)
correspond to homogeneity lengths,
which reflect the space-time scales of
emission process



Pure HKM: spectra+interferometry(HBT) radii for 200A GeV RHIC

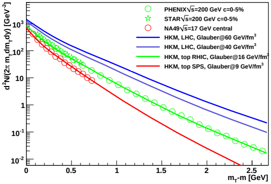


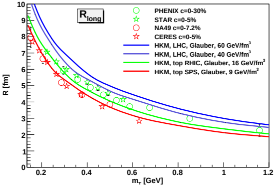
The transverse momentum spectra of negative pions and negative kaons in HKM model; the interferometry radii and R_{out}/R_{side} ratio for $\pi^-\pi^-$ pairs and mixture of K^-K^- and K^+K^+ pairs. The experimental data for 200A GeV collisions are taken from the STAR and PHENIX Collaborations.

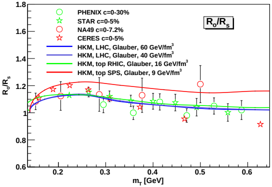
- Glauber IC:
  $\epsilon_0 = 16.5$ GeV/fm^3
  $<\epsilon>= 11.7$ GeV/fm^3
  $<v_T>= 0.224$
- CGC IC:
  $\epsilon_0 = 19.5$ GeV/fm^3
  $<\epsilon>= 13.2$ GeV/fm^3
  $<v_T>= 0.208$

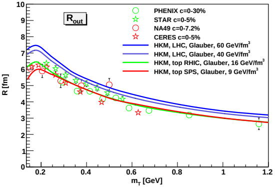
Bigger $<v_T>$ accumulates viscosity effects, EbE, etc

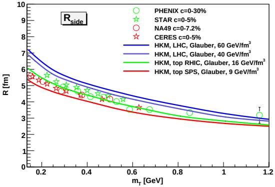
Pure HKM: top SPS + top RHIC + LHC predictions









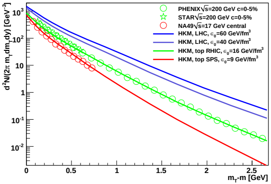


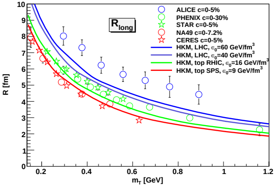
Model parameters:

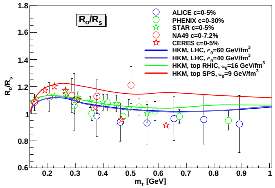
	ϵ_0	α
top SPS	9	0.194
RHIC	16.5	0.28
LHC(1)	40	0.28
LHC(2)	60	0.28

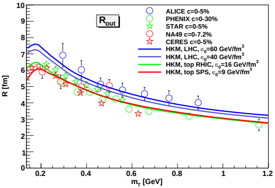
Initial energy density estimate for LHC is taken from CGC model (T. Lappi).

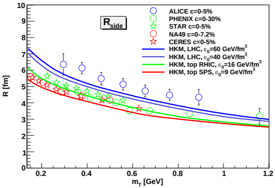
...and ALICE data











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	ε_0	α
top SPS	9	0.194
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- R_{long} is underestimated by 20% at LHC
- Reason: HKM describes a gradual decay of the system which evolves hydrodynamically until fairly large times.
- Growth of interferometry volume at LHC can be explained by protracted non-equilibrated(non-hydrodynamic) hadron phase.

Hybrid HKM = HKM + UrQMD

Goal: initial conditions for Boltzmann cascade (UrQMD) from hydro-kinetic approach.

- switching hypersurface (e.g. $\tau =$const)
- DFs from HKM:

$$f_i(\tau,\theta,\mathbf{r}_T,\mathbf{p}_T) = f_i^{l.eq.}(x^{(\tau_0)}(\tau),\mathbf{p}_T)\exp\left(-\int\limits_{\tau_0}^{\tau}\tilde{R}_i(x^{(s)}(\tau),\mathbf{p}_T)ds\right) + \int\limits_{\tau_0}^{\tau}d\lambda\left[f_i^{l.eq.}(x^{(\lambda)}(\tau),\mathbf{p}_T)\tilde{R}_i(...)\right.$$

$$\left. + \tilde{G}_i^{decay}(x^{(\lambda)}(\tau),\mathbf{p}_T) - L_i^{decay}(x^{(\lambda)}(\tau),\mathbf{p}_T)\right]\exp\left(-\int\limits_{\lambda}^{\tau}\tilde{R}_i(x^{(s)}(\tau),\mathbf{p}_T)ds\right)$$

$$p^0\frac{d^3N}{d^3p} = \int d\sigma_\mu p^\mu f(x,p) \quad (4)$$

- average particle multiplicities $< N_i >$ and maximum value of (4) over phase-space
- in each event, particle multiplicities are Poisson-distributed with mean $< N_i >$
- particle and coordinate generation: acceptance-rejection method based on distribution (4)

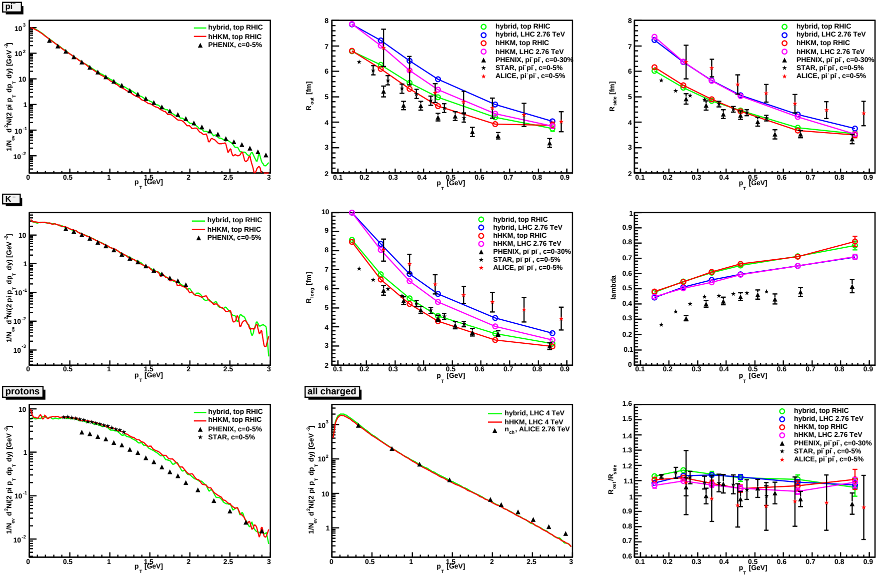
Hybrid HKM: parameters

- $\varepsilon_0 \Leftarrow$ from dN_{charged}/dy for top RHIC and $dN_{\text{charged}}/d\eta$ for LHC 2.76 TeV collision energies at mid-rapidity
  $\varepsilon_0 = 15$ GeV/fm^3 ($\langle\varepsilon_0\rangle = 10.6$ GeV/fm^3) for top RHIC energy
  $\varepsilon_0 = 40$ GeV/fm^3 ($\langle\varepsilon_0\rangle = 28.2$ GeV/fm^3) for LHC 2.76 TeV
- $\alpha \Leftarrow$ from effective temperature of hadron spectra
- the magnitude of initial transverse flow $\alpha = 0.18$ does not change from top RHIC to LHC case

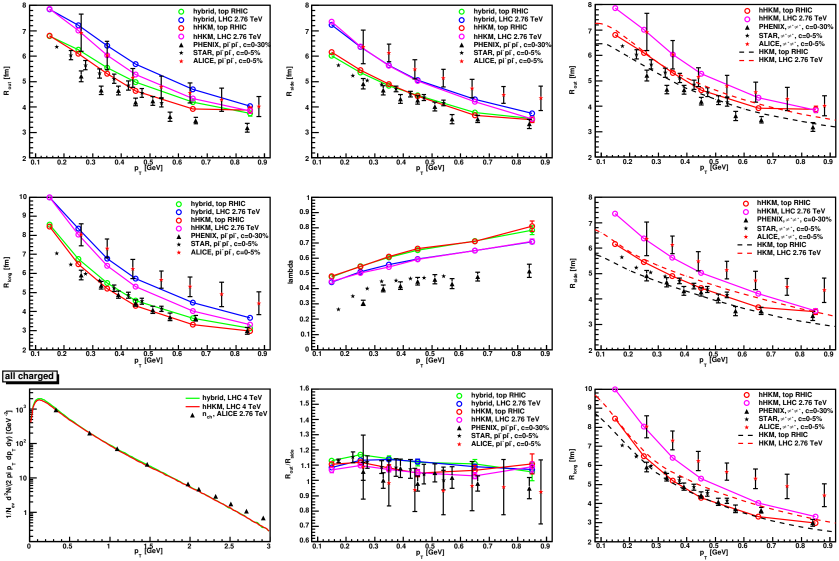
To demonstrate the difference between hHKM and *hybrid* approaches, three cases were studied:

1. switching from HKM to UrQMD on "isochrone" ($\tau = const$) corresponding to $T(r_T = 0, \tau) = T_{ch}$ according to non-equilibrium DFs from hydrokinetic model.
2. switching to UrQMD on the isotherm corresponding to chemical freeze-out temperature (*hybrid model*).
3. switching from HKM to UrQMD on the isotherm $T = 130$ MeV.

Hybrid HKM: results



Hybrid HKM versus HKM



Conclusions

- The first results of calculations in **hybrid hydro-kinetic approach**, where particle interactions at the latest stage of collision are treated with transport code (UrQMD) are presented.
- Energy dependence of HBT-radii is much improved compared to the previous results from pure hydro-kinetic model [**?**] due to UrQMD code which is essential for the late, highly nonequilibrium stage of matter expansion. The results indicate notable influence of the last, non-equilibrated stage of evolution to space-time scales at LHC energy.
- HHKM results are compared to the *hybrid model* calculations, where the hydrodynamic evolution is switched directly to UrQMD at chemical freeze-out hypersurface. It is found that, whereas p_T-dependence of R_{long} is better reproduced in a case of hybrid model, $R_{\text{out}}/R_{\text{side}}$ ratio favors hHKM procedure.
- The lack of reproduction of Ro/Rs ratio in hybrid model can be caused by the inconsistenies of hybrid approach: casuality problems at non-space-like sectors of freeze-out hypersurface and, possibly, application of transport code to a very dense system.

Thank you!

Extra slides

Hydrokinetics: relaxation time approximation for emission function

For 1-component system:

$$\frac{p^\mu}{p^0}\frac{\partial f_i(x,p)}{\partial x^\mu} = G_i(x,p) - L_i(x,p)$$

$$f(t,\vec{x},p) = f(\bar{x}_{t\to t_0},p)\mathscr{P}_{t_0\to t}(\bar{x}_{t\to t_0},p) + \int\limits_{t_0}^{t} \underbrace{G((\bar{x}_{t\to s},p))\mathscr{P}_{s\to t}(\bar{x}_{t\to s},p)}_{S(\bar{x}_{t\to s},p)} ds$$

$$\frac{d^3N}{d^3p}(t) = n(t,p) = \int d^3x f(t,x,p) \qquad \bar{x}_{t\to s} = (s, \vec{x} - \frac{\vec{p}}{p^0}(t-s))$$

$$L_i(x,p) = R_i(x,p) f_i(x,p)$$

$$R(x,p) \approx R_{l.eq.}(x,p), G \approx R_{l.eq.}(x,p) f_{l.eq.}(x,p). \tag{5}$$

$$\frac{p^\mu}{p_0}\frac{\partial f(x,p)}{\partial x^\mu} = -\frac{f(x,p) - f^{\mathrm{l\ eq}}(x,p)}{\tau_{\mathrm{rel}}(x,p)}.$$

Approximate solution:

$$f = f^{\mathrm{l\ eq}}(x,p) + g(x,p), \quad \text{äå}$$

$$g(x,p) = -\int\limits_{t_0}^{t} \frac{df^{\mathrm{l\ eq}}(t',\mathbf{r}-\frac{\mathbf{p}}{p_0}(t-t'),p)}{dt'} \exp\left\{-\int\limits_{t'}^{t}\frac{1}{\tau_{\mathrm{rel}}(s,\mathbf{r}-\frac{\mathbf{p}}{p_0}(t-s),p)}ds\right\} dt'.$$

Hydrokinetics: general formalism

$$\partial_\nu T^{\nu\beta}[f^{\mathrm{l\ eq}}] = G^\beta[g], \quad (6)$$

where

$$G^\beta[g] = -\partial_\nu T^{\nu\beta}[g]. \quad (7)$$

for particle number:

$$\partial_\nu n^\nu[f^{\mathrm{l\ eq}}] = S[g], \quad (8)$$

where

$$S[g] = -\partial_\nu n^\nu[g]. \quad (9)$$

To find the approximate solution of BE,
1. Solve ideal hydro equations:

$$\partial_\nu T^{\nu\mu}[f^{\mathrm{l\ eq}}] = 0, \quad (10)$$

$$\partial_\nu n^\nu[f^{\mathrm{l\ eq}}] = 0, \quad (11)$$

2. Use the values obtained to calculate the deviations from equilibrium $g(x,p)$, and use them to solve (6,8) in the following approximation:

$$\partial_\nu T^{\nu\beta}[f^{\mathrm{l\ eq}}(T, u_\mu, \mu)] = G^\beta[T_{\mathrm{id}}, u_\mu^{\mathrm{id}}, \mu_{\mathrm{id}}, \tau_{\mathrm{rel}}^{\mathrm{id}}], \quad (12)$$

$$\partial_\nu n^\nu[f^{\mathrm{l\ eq}}(T, u_\mu, \mu)] = S[T_{\mathrm{id}}, u_\mu^{\mathrm{id}}, \mu_{\mathrm{id}}, \tau_{\mathrm{rel}}^{\mathrm{id}}], \quad (13)$$